A CLASS OF OPTIMAL DESIGNS FOR CULTIVATORS' FIELD TRIALS

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1. INTRODUCTION

Conducting experiments on cultivators' fields is an important activity of experimentation being undertaken by I.C.A.R. since the past two decades. In some of the fertilizer trials conducted on cultivators' fields in the recent years, the aim has been to study the crop responses of various level combinations of nitrogen, phosphorus and potassium. In all these trials the restricting factor is the size of the design which usually is kept up to ten distinct level combinations.

Consider the factorial $s_1 \times s_2 \times \ldots \times s_m$ with m factors such that the i-th factor is experimented with s_i levels. If it is desired to experiment with only n runs where n is a subset of $N(=s_1 \times s_2 \times \ldots \times s_m)$ then such a design can be chosen in $\binom{N}{n}$ ways. Out of $\binom{N}{n}$ possible designs (choices), the best choice is known to be one which is optimal in some sense (for example, see [3]).

In this paper we have examined all possible choices for a class of designs needed for cultivators' field trials. These have been studied on the basis of various known optimality criteria like A-, D-, E-, and in term of a new criterion *viz.*, C-optimality, proposed in the following section. The best design is seen to be much more efficient than the one currently being used by I.C.A.R. for cultivators' field trials. In fact, out of 81 possible designs, the design in current use has the rank 49 in terms of C-optimal criterion.

2. PRELIMINARIES

Consider the experimental design model

$$\underline{Y} = \underline{X}\underline{\beta} + \underline{e},$$

where $\underline{Y}$ is the $n \times 1$ vector of observations, $\underline{X}$ is the design martix of order $n \times p$, $\underline{\beta}$ is the $p \times 1$ vector of parameters (which in case of factorial model are the effects having single degree of freedom) and $\underline{e}$

is the $n \times 1$ vector of error components assumed to be normal with mean 0 and with covariance matrix $\sigma^2 \underline{I}$. The best linear unbiased estimates $\underline{\hat{\beta}}$ of $\underline{\beta}$ are the least squares estimates given by $\hat{\underline{\beta}} = (\underline{X}' \underline{X})^- \underline{X}' \underline{Y}$ and variance-covariance matrix of $\hat{\underline{\beta}}$ is $(\underline{X}' \underline{X})^- \sigma^2$. Here $(\underline{X}' \underline{X})^-$ is a generalised inverse of $\underline{X}' \underline{X}$ if $\underline{X}' \underline{X}$ is not of full rank, otherwise $(\underline{X}' \underline{X})^-$ is replaced by the unique inverse $(\underline{X}' \underline{X})^{-1}$. The A, D and E-optimality criteria as elaborated in [4] on page 315 are presented below:

Definition 1. Of the class of all $n \times p$ designs, the design $\overset{*}{D}$ is A-optimal if it has the least value for the trace of $(\underline{X}' \underline{X})^{-1}$.

Definition 2. Of the class of all $n \times p$ designs, the design $\overset{*}{D}$ is D-optimal if it has the minimum value of the determinant of $(\underline{X}' \underline{X})^{-1}$.

Definition 3. Of the class of all $n \times p$ designs, the design $\overset{*}{D}$ is E-optimal if it has least value for λ_{max}, where λ_{max}. is the maximum eigen value of $(\underline{X}' \underline{X})^{-1}$.

It is known (*e.g.*, see [5]) that all these criteria need not provide the same design, *i.e.*, the A-optimal design need not be D-optimal, the D-optimal need not be E-optimal, and so on. One can, therefore, look for another criterion which may reflect in it all the three criteria described above.

Defintion 4. Of the class of all $n \times p$ designs, the design $\overset{*}{D}$ is C-optimal if it has the minimum value of the sum of percentage departures from the optimum value for the three criteria described earlier.

Thus, if D_{ij} represents the % departure from the optimum point for the i-th design ($i = 1, 2 \ldots, \binom{N}{n}$) and j-th optimality criterian ($j = 1, 2, 3$ for A-, B- and E-optimality respectively), then the C-optimal design is the one for which ϕ_i is minimum where

$$\phi_i = \sum_{j=1}^{3} D_{ij}$$

3. Class of Optimal Designs for Cultivators' Field Trials

We first describe the important features of the design currently being used by the I C.A.R. on cultivators' fields. The design consists of the following ten treatment combinations :

N	P	K
0	0	0
2	0	0
3	3	3
2	0	2
2	2	0
2	1	2
2	2	2
2	2	1
2	2	3
2	3	2

where 0, 1, 2 and 3 are the levels of each factor. Here 0 indicates no application of fertilizer and 2 as the base level. The object of the experiment is to study the responses of N, P and K (each at four levels) with a view to formulate fertilizer recommendations for different agro-climatic conditions in the country.

The treatment combinations 000, 200 and 333 are included in the above design in order to study the responses of average effect, effect of nitrogen at base level and the combined effect of higher levels of N, P and K. In the remaining seven treatment combinations the nitrogen is at the base level and the levels of P and K vary to study their responses for different levels at the base level of nitrogen.

Hence the problem reduces to choose a set of seven treatment combinations from the 4^2 factorial experiment (P and K each at four levels).

In order to study the responses of P and K at the base level the treatment combinations 20 and 02 have also been fixed and the remaining five treatment combinations have to be chosen out of the nine treatment combinations 11, 12, 13, 21, 22, 23, 31, 32 and 33, so that response curves due to P and K can be fitted at the base level of nitrogen. Thus, we get in all, 126 sets (designs) each with five treatment combinations. Any one set from these 126 sets combined with two treatment combinations 20, and 02 gives a set of seven treatment combinations.

Now our aim is to study all the 126 sets in terms of various optimality criteria elaborated in the foregoing sections. The vector $\underline{\beta}$ consists of the general mean and all the main effects each with single degree or freedom (when the assumption that all higher order interactions are negligible is justified). Each main-effect has 3 *d.f.* which can further be partitioned into linear, quadratic and cubic effects each with 1 *d.f.*

Out of 126 sets, 45 sets were such for which the vector $\underline{\beta}$ could not be estimated as the value of the determinant of $(\underline{X}' \underline{X})^{-1}$ was zero. Remaining 81 sets have been explored and the values of the determinant of $(\underline{X}' \underline{X})^{-1}$, maximum eigen value and trace of $(\underline{X}' \underline{X})^{-1}$ computed.

Set Nos. 3, 7, 9, 37, 48, 60, 69, 72, 117 and 119 turn out as the *A*-optimal designs, set no. 79 is optimal with respect to *D*-optimality and set nos. 117 and 119 are the *E*-optimal designs. All the sets have been ranked according to the three optimality criteria and first five ranks have been tabulated for each optimality criterion in tables 1, 2 and 3. Set no. 92, which is the design currently being used for cultivators' field trials has also been included for the sake of comparison. This design ranks 8th in term of *A*-optimality criterion, 19th in term of *D*-optimality and 21st in term of E-optimality criterion. The

TABLE 1

(A-Optimal)

Rank	*Set Nos.*	*Trace of* $(\underline{X}' \underline{X})^{-1}$
1	3, 7, 9, 37, 48, 60, 69, 72, 117, 119	1.050
2	30, 39, 46, 49, 51, 67, 75, 78, 85, 110	1.100
3	8, 31, 56, 58, 73, 84, 91, 93, 96, 104, 106, 108	1.125
4	79,	1.150
5	5, 24, 74, 77, 88, 95, 100, 102, 111, 115	1.175
8	92	1.320

TABLE 2

(D-Optimal)

Rank	Set Nos.	Value of Det. of $(\underline{X}'\,\underline{X})^{-1}$
1	79	$0.24414201 \times 10^{-7}$
2	7, 37	$0.24414204 \times 10^{-7}$
3	21	$0.24414219 \times 10^{-7}$
4	52	$0.24414230 \times 10^{-7}$
5	11	$0.24414238 \times 10^{-7}$
19	92	$0.24414276 \times 10^{-7}$

TABLE 3

(E-Optimal)

Rank	Set Nos.	Maximum eigen value
1	117, 119	0.3837
2	9, 48, 60, 69	0.4114
3	30, 39	0.4350
4	31, 58, 93	0.4469
5	98, 104, 108	0.4564
21	92	0.8137

design is thus not a good choice. As none of the sets is optimal with respect to all the A, D-and E-optimality, we go for C-optimality described earlier. The first ten ranking sets, along-with the set no. 92 have been presented in table 4. It may be seen that the set no. 92 is very poor performer. Set no. 117 comes out to be the C-optimal design. The seven treatment combinations for this design are as given below:

$\underline{P}$	$\underline{K}$
0	2
2	0
1	3
2	2
2	3
3	1
3	3

TABLE 4

(C-Optimal)

Rank	*Set Nos.*	$\sum_{j=1}^{3} D_{ij}$
1	117	28.35
2	7,37	37.34
3	119	47.76
4	60	49.68
5	51	55.80
6	69	59.13
7	67	64.75
8	31	66.38
9	3	71.16
10	72	71.17
49	92	166.18

However, this design may not be practicable in terms of cost considerations as it involves higher levels of P and K. Amongst the top ten ranking sets the set no. 3 in table 4 may be preferable as it involves lower levels of P and K. The level combinations of this design are :

P	K
0	2
2	0
1	1
1	2
1	3
2	1
3	1

This design is superior to 92 not only in terms of C-optimality, but also in terms of all the three A-, D-and E-optimality criteria. It also performs better in terms of cost considerations as it involves lower levels of P and K. Considering the enormous number of such experiments being conducted all over the country it is all the more necessary to prefer design 3 to 92. The complete design for set 3 is given below :

N	*P*	*K*
0	0	0
2	0	0
3	3	3
2	2	0
2	0	2
2	1	1
2	1	2
2	1	3
2	2	1
2	3	1

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